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On superquadratic Dirac equations on compact spin manifolds

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1 Introduction

In this note, we report our recent work about Morse theory for superquadratic Dirac equations on compact spin manifolds. The details will appear in [18].

Let (M, g, ρ) be an m-dimensional compact Spin manifold, where g is a Riemannian metric on M, $\rho : P_{Spin}(M) \to P_{SO}(M)$ is a spin structure on M. We denote by $\mathbb{S}(M) = P_{Spin}(M) \times_{\sigma} \mathbb{S}_m \to M$ the spinor bundle. It is a vector bundle associated to $P_{Spin}(M) \to M$ via the fundamental spin representation $\sigma : Spin(m) \to \mathrm{Aut}(\mathbb{S}_m)$. The Dirac operator $\mathsf{D}_g : C^{\infty}(M, \mathbb{S}(M)) \to C^{\infty}(M, \mathbb{S}(M))$ is define by

$$\begin{aligned}\mathsf{D}_g := c \circ \nabla : C^{\infty}(M, \mathbb{S}(M)) &\xrightarrow{\nabla} C^{\infty}(M, T^*M \otimes \mathbb{S}(M)) \\ &\cong C^{\infty}(M, TM \otimes \mathbb{S}(M)) \xrightarrow{c} C^{\infty}(M, \mathbb{S}(M)),\end{aligned}$$

where ∇ is the canonical lift of the Levi-Civita connection on $P_{SO}(M)$ via the double covering $P_{Spin}(M) \to P_{SO}(M)$ and c is the Clifford multiplication.

We consider nonlinear Dirac equations of the following form:

$$\mathsf{D}_g \psi = h(x, \psi) \quad \text{on } M, \tag{1.1}$$

where $h : \mathbb{S}(M) \to \mathbb{S}(M)$ is a fiber preserving map of the form $h(x, \psi) = \nabla_{\psi} H(x, \psi)$, the vertical gradient of H (the dual of $d_{\psi} H$ w.r.t the metric on $\mathbb{S}(M)$) and $H = H(x, \psi)$ a smooth function on $\mathbb{S}(M)$.

Eq (1.1) has a variational structure: ψ is a solution to (1.1) if and only if ψ is a critical point of $\mathcal{L}_H$ defined by

$$\mathcal{L}_H(\psi) = \frac{1}{2} \int_M \langle \psi, \mathsf{D}_g \psi \rangle \, d\mathrm{vol}_g - \frac{1}{p+1} \int_M H(x, \psi) \, d\mathrm{vol}_g.$$

We give in the following two examples which partially motivate our study of equations of the form (1.1).

Example 1: Dirac harmonic maps (Spinorial version of SUSY σ-model)
This model was first introduced by [10], [9]. Let (Σ, g) be a Riemann surface and (N, h) a Riemannian manifold. In this model, we consider a pair of two fields $\phi \in C^\infty(\Sigma, N)$ and $\psi \in C^\infty(\Sigma, \mathbb{S}(\Sigma) \otimes \phi^* TN)$. In components, we write $\psi = \psi^k \otimes \frac{\partial}{\partial y^k}(\phi)$, where y^k a local coordinate system on N and $\psi_k \in C^\infty(\mathbb{S}(\Sigma))$.

The action functional for supersymmetric Dirac-harmonic maps is defined by

$$\begin{aligned}\mathcal{L}(\phi, \psi) =& \frac{1}{2} \int_\Sigma |d\phi|^2 \, d\mathrm{vol}_g + \frac{1}{2} \int_\Sigma \langle \psi, \mathsf{D}_\phi \psi \rangle \, d\mathrm{vol}_g \\ &- \frac{1}{12} \int_\Sigma R_{ikjl}(\phi) \langle \psi^i, \psi^j \rangle \langle \psi^k, \psi^l \rangle \, d\mathrm{vol}_g,\end{aligned}$$

where $\mathsf{D}_\phi = c \circ \nabla^\phi$ is the Dirac operator associated to the natural connection ∇^ϕ on $\mathbb{S}(\Sigma) \otimes \phi^* TN$ and R_{ijkl} the curvature tensor of (N, h).

The Euler-Lagrange equation for the action takes the following form:

$$\tau^m(\phi) - \frac{1}{2} R^m_{lij}(\phi) \langle \psi^i, \nabla \phi^l \cdot \psi^j \rangle + \frac{1}{12} g^{mp} R_{ikjl;p}(\phi) \langle \psi^i, \psi^j \rangle \langle \psi^k, \psi^l \rangle = 0, \tag{1.2}$$

$$\mathsf{D}_\phi \psi^m = \frac{1}{3} R^m_{jkl}(\phi) \langle \psi^j, \psi^l \rangle \psi^k, \tag{1.3}$$

where $\tau(\phi) = \mathrm{tr} \nabla d\phi$.

The main characteristic of this problem is the following:

• $\mathcal{L}$ is conformally invariant.

As a result, (1.2), (1.3) are conformally invariant equations and depend only on the confomal structure of (Σ, g).

• $\mathcal{L}$ is quartic in ψ.

Combined with the fact that $H^{1/2}(\Sigma) \subset L^4(\Sigma)$ is continuous, but not compact, we have:

• The variational problem associated to $\mathcal{L}$ is non-compact and strongly indefinite.

At present, there are no general existence results for (1.2), (1.3) from a variational point of view. Note, however, that there is a variational theory for the 1-dimensional case, the so-called Dirac-geodesics, see [16].

Example 2: Spinorial Yamabe type equations

Let (M, g, ρ) be a compact spin manifold. We assume $H \in C^\infty(M)$ is given. We consider the following action functional:

$$\mathcal{L}(\psi) = \frac{1}{2} \int_M \langle \psi, \mathsf{D}_g \psi \rangle \, d\mathrm{vol}_g - \frac{m-1}{2m} \int_M H(x) |\psi|^{\frac{2m}{m-1}} \, d\mathrm{vol}_g.$$

The Euler-Lagrange equation of this action is

$$\mathsf{D}_g \psi = H(x) |\psi|^{\frac{2}{m-1}} \psi. \tag{1.4}$$

The equation (1.4) is related to the existence of conformal immersion $M \to \mathbb{R}^{m+1}$ with mean curvature H. The following are main characteristic of this problem:

• $\mathcal{L}$ is conformally invariant.

Thus, (1.4) is a conformally invariant equation and depends only on the conformal structure of (M, g).

• $|\psi|^{\frac{2m}{m-1}}$ is the critical power.

That is, $H^{1/2}(M) \subset L^{\frac{2m}{m-1}}(M)$ is continuous, but not compact. As a result, the associated variational problem is critical and strongly indefinite.

Partial existence results were previously established by [21] and [17]. However, the problem remains widely open in general.

Both of examples 1,2 are **non-compact, strongly indefinite** variational problems. We want to establish a general variational framework to treat such problems. Some compactness issues were treated in [15]. In the work [18], we focus on the indefinite variational character of the problem. In this direction, we are especially interested in two topics:

(1) Relative Morse indices and its connection with compactness property.

(2) Construction and computation of Morse-Floer homology of $H^{1/2}(M, \mathbb{S}(M))$ associated to $\mathcal{L}_H$.

2 A superquadratic subcritical problem

We first introduce functional setting of the problem. A natural function space is $H^{1/2}$-spinors on M denoted by $\mathcal{H}^{1/2}(M) := H^{1/2}(M, \mathbb{S}(M))$. It is defined as:

$$\psi \in \mathcal{H}^{1/2}(M) \quad \Leftrightarrow \quad \psi \in L^2(M),\ |\mathsf{D}_g|^{1/2}\psi \in L^2(M).$$

$\mathcal{H}^{1/2}(M)$ is a Hilbert space with the following inner product

$$(\psi, \varphi)_{H^{1/2}} := (|\mathsf{D}_g|^{1/2}\psi, |\mathsf{D}_g|^{1/2}\varphi)_{L^2} + (\psi, \varphi)_{L^2}.$$

We have the Sobolev embedding:

$$H^{1/2}(M) \subset L^{p+1}(M) \quad \text{for } 0 \le p \le \frac{m+1}{m-1}.$$

The embedding is compact for $0 \le p < \frac{m+1}{m-1}$, but not for $p = \frac{m+1}{m-1}$. $p + 1 = \frac{2m}{m-1}$ is called the critical exponent.

For H, we assume the following condition:

$$|H(x, \psi)| \le C(1 + |\psi|^{p+1}) \tag{2.1}$$

for some $1 < p < \frac{m+1}{m-1}$. Under the condition (2.1), $\mathcal{L}_H$ is a **subcritical** functional.

We further assume $H(x, \psi)$ is C^2 and satisfies

$$|d^2_{\psi\psi} H(x, \psi)| \le C(1 + |\psi|^{p-1}). \tag{2.2}$$

Under the condition (2.2), $\mathcal{L}_H$ is C^2 on $\mathcal{H}^{1/2}(M)$. Note that (2.1) follows from (2.2).

We also assume the following **superquadratic** condition:

$$2H(x,\psi) + C_1|\psi|^{p+1} - C_2 \le \langle \psi, H_\psi(x,\psi)\rangle. \tag{2.3}$$

A model example:

$$H(x,\psi) = \frac{1}{p+1} H(x)|\psi|^{p+1},$$

$H(x) > 0$, $H \in C^0(M)$.

We want to establish a Morse theory for $\mathcal{L}_H$ on $\mathcal{H}^{1/2}(M)$ for the class of H satisfying (2.2) and (2.3).

3 Relative Morse indices and compactness

To do Morse theory, we first need to define Morse index for a critical point of the functional. Classically, Morse index (co-index) at a critical point $\psi \in \mathcal{H}^{1/2}(M)$ is defined as the dimension of the maximal subspace of $\mathcal{H}^{1/2}(M)$ on which $d^2\mathcal{L}_H(\psi) < 0$ (> 0), where

$$d^2\mathcal{L}_H(\psi)(\varphi,\varphi) = \int_M \langle \varphi, \mathsf{D}_g\varphi\rangle\, d\mathrm{vol}_g - \int_M \langle H_{\psi\psi}(x,\psi)\varphi,\varphi\rangle\, d\mathrm{vol}_g.$$

Equivalently, it is the dimension of the negative eigenspaces of $\mathsf{D}_g - H_{\psi\psi}(x,\psi)$.

Note that $\mathsf{Spec}(\mathsf{D}_g)$ is unbounded from below and above. This implies that the Morse index and co-index are $+\infty$ at any critical point. Thus the classical Morse theory does not make sense for $\mathcal{L}_H$ on $\mathcal{H}^{1/2}(M)$. We need a renormalized version of the classical Morse theory.

3.1 Relative Morse indices

To construct right Morse theory, we need renormalized Morse indices. In the following, we introduce three well-known definitions of such renormalized Morse indices.

1. Relative Morse index as relative dimension

The idea of this definition is to compare the negative space of $d^2\mathcal{L}_H(\psi)$ with some fixed subspace. Let $V, W \subset \mathcal{H}^{1/2}(M)$ be subspaces. Following [1], [2], we say V, W commensurable if $P_V - P_W$ is compact, where $P_V : \mathcal{H}^{1/2}(M) \to V$ is the orthogonal projection onto V. P_W is defined similarly. For such commensurable subspaces V, W, we define the relative dimension $\dim(V,W)$ as

$$\dim(V,W) = \dim(V \cap W^\perp) - \dim(V^\perp \cap W).$$

Note that the $H^{1/2}$-self-adjoint realization of $d^2\mathcal{L}_H(\psi)$ is given by $d^2\mathcal{L}_H(\psi) = (|\mathsf{D}_g|+1)^{-1}\mathsf{D}_g - (|\mathsf{D}_g|+1)^{-1}H_{\psi\psi}(x,\psi)$.

We define $E_H^-(\psi) = E^-(d^2\mathcal{L}_H(\psi))$, the negative eigenspace of $d^2\mathcal{L}_H(\psi)$. We also define $\mathsf{D}_\lambda = \mathsf{D}_g - \lambda$ ($\lambda \in \mathbb{R}$) and $E_\lambda^- = E^-((|\mathsf{D}_g| + 1)^{-1}\mathsf{D}_\lambda)$.

With these definitions, we give the following

Definition 1 *λ-relative Morse index of $\mathcal{L}_H$ at $\psi \in \mathcal{H}^{1/2}(M)$ is defined as*

$$m_\lambda(\psi) := \dim(E_H^-(\psi), E_\lambda^-).$$

Note that since $d^2\mathcal{L}_H(\psi) - (|\mathsf{D}_g| + 1)^{-1}\mathsf{D}_\lambda$ is compact, the above definition is well-defined, i.e., $m_\lambda(\psi) \in \mathbb{Z}$.

2. Relative Morse index as spectral flow

By $d^2\mathcal{L}_H(\psi)(\varphi,\varphi) = ((\mathsf{D}_g - H_{\psi\psi}(x,\psi))\varphi, \varphi)_{L^2}$, $d^2\mathcal{L}_H(\psi) = \mathsf{D}_g - H_{\psi\psi}(x,\psi) : L^2(M,\mathbb{S}(M)) \to L^2(M,\mathbb{S}(M))$ is the L^2-self-adjoint realization of $d^2\mathcal{L}_H(\psi)$.

We set $A_\psi := H_{\psi\psi}(x,\psi) : \mathbb{S}(M) \to \mathbb{S}(M)$. It is a symmetric endmorphism of $\mathbb{S}(M)$. We define $\mathsf{D}_A := \mathsf{D}_g - A$ for $A \in \mathcal{A} = L^\infty(M, \mathrm{Sym}(\mathbb{S}(M)))$.

Let us consider a continuous path $\{\mathsf{D}_{A_t}\}_{t\in[0,1]}$ connecting D_λ and D_{A_ψ}. It is a fact that the eigenvalues of a generic path $\{\mathsf{D}_{A_t}\}_{t\in[0,1]}$ is simple. The spectral flow $\mathsf{sf}\{\mathsf{D}_{A_t}\}_{t\in[0,1]}$ is defined as

$$\begin{aligned}&\mathsf{sf}\{\mathsf{D}_{A_t}\}_{t\in[0,1]}\\ &= \text{the number of eigenvalues flowing from negative to positive}\\ &- \text{the number of eigenvalues flowing from positive to negative.}\end{aligned}$$

With these, we give

Definition 2 *Relative Morse index $\mu_\lambda(\psi)$ is defined as*

$$\mu_\lambda(\psi) = -\mathsf{sf}\{\mathsf{D}_{A_t}\}_{t\in[0,1]}.$$

3. Relative Morse index as Fredholm index

We consider the negative gradient flow connecting two critical points $\mathsf{x}, \mathsf{y} \in \mathrm{crit}(\mathcal{L}_H) := \{\psi \in \mathcal{H}^{1/2}(M) : d\mathcal{L}_H(\psi) = 0\}$:

$$\frac{\partial\psi}{\partial t} = -\nabla_{1/2}\mathcal{L}_H(\psi), \quad \psi(-\infty) = \mathsf{x},\ \psi(+\infty) = \mathsf{y}, \tag{3.1}$$

where $\nabla_{1/2}\mathcal{L}_H(\psi) = (|\mathsf{D}_g| + 1)^{-1}\mathsf{D}_g - (|\mathsf{D}_g| + 1)^{-1}\nabla_\psi H(x,\psi)$ is the $H^{1/2}$-gradient of $\mathcal{L}_H$.

It is a fact that (3.1) is Fredholm if x, y are non-degenerate, see [2]. The Fredholm index of (3.1) is the Fredholm index of the linearization:

$$\frac{\partial u}{\partial t} = -d\nabla_{1/2}\mathcal{L}_H(\psi)u, \quad u(-\infty) = 0, \quad u(+\infty) = 0.$$

Theorem 1 *The Fredholm index of (3.1) depends only on $d\nabla_{1/2}\mathcal{L}_H(\mathsf{x})$ and $d\nabla_{1/2}\mathcal{L}_H(\mathsf{y})$. We denote it by $\mu(x,y)$.*

For the proof of the above theorem, see [6], [4]. The three indices $m_\lambda(\psi)$, $\mu_\lambda(\psi)$ and $\mu(x,y)$ are related as follows:

Theorem 2 *Assume $\psi \in crit(\mathcal{L}_H)$ is non-degenerate and $\lambda \in \mathbb{R} \setminus \mathsf{Spec}(\mathsf{D}_g)$. The following hold:*

(1) $m_\lambda(\psi) = \mu_\lambda(\psi)$

(2) $\mu(x,y) = m_\lambda(x) - m_\lambda(y)$

For the proof of (1), see [19]. For the proof of (2), see [6], [4].

3.2 A compactness theorem via relative Morse indices

We assume that $\mathcal{L}_H$ is Morse on $\mathcal{H}^{1/2}(M)$ and $\mathbb{F}$ is a field. We define a graded group $\{C_p(\mathcal{L}_H, \mathcal{H}^{1/2}(M))\}_{p\in\mathbb{Z}}$ by

$$C_p(\mathcal{L}_H, \mathcal{H}^{1/2}(M)) = \bigoplus_{\psi \in \mathrm{crit}_p(\mathcal{L}_H)} \mathbb{F}\langle\psi\rangle,$$

where $\mathrm{crit}_p(\mathcal{L}_H, \mathcal{H}^{1/2}(M)) = \{\psi \in \mathrm{crit}(\mathcal{L}_H) : m_\lambda(\psi) = p\}$.

Since $\mathcal{H}^{1/2}(M)$ is not compact, $C_p(\mathcal{L}_H, \mathcal{H}^{1/2}(M))$ is not necessarily finitely generated. But, it is the case for some class of H including lower order perturbations of the model example given in §2. Our first result is **a compactness result under the relative Morse index bounds**. We only state it for the model case. A more general result will be found in [18].

Theorem 3 *Let $m \geq 3$. Assume that $H(x,\psi) = \frac{1}{p+1}H(x)|\psi|^{p+1}$, where $H \in C^0(M)$ with $H > 0$ on M and $1 < p < \frac{m+1}{m-1}$. The following assertions are equivalent for a sequence of solutions $\{\psi_n\}_{n=1}^\infty \subset \mathcal{H}^{1/2}(M)$ to (1.1).*

(1) $\sup_{n\geq 1} \mathcal{L}_H(\psi_n) < +\infty$.

(2) $\sup_{n\geq 1} m_\lambda(\psi_n) < +\infty$.

Remark 1 *(1) In [8], Bahri-Lions considered equations of the form $-\Delta u = a(x)|u|^{p-1}u$ in $\Omega \subset \mathbb{R}^m$ under the 0-Dirichlet boundary condition and obtained similar result, where the Morse index is the classical one.*

• *Angenent-van der Vorst [7] considered equations $-\Delta u = a(x)|v|^{p-1}v$, $-\Delta v = b(x)|u|^{q-1}u$ in $\Omega \subset \mathbb{R}^m$ under the 0-Dirichlet boundary conditions. In this case, these equations arise as a critical point of an indefinite functional. Thus the Morse index is the relative Morse indices. Thus it is related with our work. In fact, we owe much to their ideas in our proof of Theorem 3. In this case, however, by the special structure of the equation, we can eliminate one of the functions, for example v, from the 2nd equation, and the problem is reduced to an ordinary (classical) index problem*

of a single function. This approach can not be applied to our problem. Thus, we will take another approach.
(2) From Theorem 3, we have that $crit_p(\mathcal{L}_H) \subset \mathcal{H}^{1/2}(M)$ is relatively compact for any $p \in \mathbb{Z}$.

3.3 Idea of the Proof of Theorem 3, the first part

(1) $\Rightarrow$ (2) is a consequence of PS and continuity of m_λ, see [14]. To prove (2) $\Rightarrow$ (1), following Bahri-Lions and Angenent-van der Vorst, we argue by contradiction. Thus, we assume that there exists $\{\varphi_n\}_{n=1}^{\infty}$, a sequence of solutions

$$\mathsf{D}_g\varphi_n = H(x)|\varphi_n|^{p-1}\varphi_n \quad \text{on } M \tag{3.2}$$

such that

$$\mathcal{L}_H(\varphi_n) \to \infty (\Leftrightarrow \|\varphi_n\|_{L^\infty(M)} \to +\infty) \tag{3.3}$$

and bounded relative Morse indices

$$m_\lambda(\varphi_n) \leq k. \tag{3.4}$$

We conformal blow-up of g: Define $(M, \rho_n^2 g := g_n)$, $\rho_n := \|\psi_n\|_{L^\infty(M)}^{p-1}$. We then have

$$(M, g_n) \underset{n\to\infty}{\to} (\mathbb{R}^m, g_{\mathbb{R}^m}) \quad \text{in } C^\infty_{\text{loc}}(\mathbb{R}^m).$$

Define $\psi_n = \rho_n^{-\frac{1}{p-1}} F(\varphi_n)$ on $\mathbb{S}(M, g_n) \to (M, g_n)$, where $F : \mathbb{S}(M,g) \to \mathbb{S}(M, g_n)$ is a fiberwise isometry.

We note the following Conformal property of the Dirac operator:

$$\mathsf{D}_{g_n}F(\varphi) = F(\rho_n^{-\frac{m+1}{2}}\mathsf{D}_g(\rho_n^{\frac{m-1}{2}}\varphi)) = \rho_n^{-1}F(\mathsf{D}_g\varphi).$$

Thus, ψ_n satisfies

$$\mathsf{D}_{g_n}\psi_n = H(x)|\psi_n|^{p-1}\psi_n \quad \text{on } (M, g_n), \tag{3.5}$$

$$\|\psi_n\|_{L^\infty(M,g_n)} = 1 \tag{3.6}$$

and

$$m_{\rho_n^{-1}\lambda}(\psi_n) \leq k. \tag{3.7}$$

By (3.5), (3.6) we have (after a further renormalization): There exists $\psi_\infty \in L^\infty(\mathbb{R}^m, \mathbb{S}(\mathbb{R}^m))$ such that $\psi_n \to \psi_\infty$ (in $L^\infty_{\text{loc}}(\mathbb{R}^m)$),

$$\mathsf{D}_{g_{\mathbb{R}^m}}\psi_\infty = H(x_\infty)|\psi_\infty|^{p-1}\psi_\infty \quad \text{on } \mathbb{R}^m, \tag{3.8}$$

$$\|\psi_\infty\|_{L^\infty(\mathbb{R}^m)} = 1. \tag{3.9}$$

By (3.7), we want to assert

$$m_0(\psi_\infty) \leq k. \tag{3.10}$$

But, $m_0(\psi_\infty)$ has not been defined yet!

• Three definitions of the relative Morse indices we have been given before depend crucially on the compactness of M. So, these do not apply to ψ_∞ because $\mathbb{R}^m$ is not compact.

To obtain (3.10) from (3.7), we need another formulation of the relative Morse index m_λ which can be applied to non-compact setting as well.

3.4 Relative Morse index $m_{\mathbb{R}^m}$ and its property

A natural requirement for the "relative Morse index" $m_{\mathbb{R}^m}$ is the following:

(I-1) $m_{\mathbb{R}^m}$ is defined for L^∞-solutions to (3.8) on $\mathbb{R}^m$.
(I-2) It is a natural extension of m_λ.
(I-3) It is (lower-semi)continuous w.r.t the $L^\infty_{\text{loc}}(\mathbb{R}^m)$-convergence (as in the previous subsection).

We will construct such index in the following. In addition to the above three properties, we show that our index has the following property **(I-4)**; in particular, it is non-trivial.

Theorem 4 ((I-4)) *Assume that $m \geq 3$ and $1 < p < \frac{m+1}{m-1}$. For any nontrivial solution $\psi \in L^\infty(\mathbb{R}^m, \mathbb{S}(\mathbb{R}^m))$ to $\mathsf{D}_{g_{\mathbb{R}^m}}\psi = |\psi|^{p-1}\psi$ on $\mathbb{R}^m$, we have $m_{\mathbb{R}^m}(\psi) = +\infty$.*

Remark 2 *(i) Theorem 4 gives a positive answer to the conjecture by Maalaoui [20].*
(ii) **(I-2)** *is in fact not necessary. It only motivates our construction of $m_{\mathbb{R}^m}$.*

Once we assume the existence of $m_{\mathbb{R}^m}$ with the properties (I-1)–(I-4), Theorem 3 easily follows: We have a contradiction from (I-3), (I-4) and (3.10):

$$+\infty = m_{\mathbb{R}^m}(\psi_\infty) \leq k.$$

□

4 A construction of the relative Morse index $m_{\mathbb{R}^m}$

4.1 A reformulation of the relative Morse index m_λ

As we have observed, we need a reformulation of the relative Morse indices which can be extended to non-compact manifolds (such as $\mathbb{R}^m$) as well. Our reformulation was inspired from the work of J. J. Dusitermaat [12] about classical mechanics.

An example from classical mechanics
We consider classical mechanics on compact manifold M. We denote by $\Lambda(M) = C^\infty(S^1, M)$ the free loop space on M. Let $L(t, q, v)$ be a smooth function on TM,

where $q \in M$ and $v \in T_qM$. It is called a Lagrangian on TM. We assume that L is convex with respect to v. For $q \in \Lambda(M)$, we define the action functional of L as $\mathcal{L}(q) = \int_{S^1} L(t, q(t), \dot{q}(t))\, dt$. For such L, there corresponds to a Hamiltonian function H on T^*M defined by $H(t,q,p) = \max_v(\langle q, v\rangle - L(t,q,v))$. H is the so called the Legendre dual of L. For the Hamiltonian H, we define the action $\mathcal{A}(x) = \int_{S^1} x^*\lambda - H(t, x(x))\, dt$ defined for a loop x on T^*M, $x(t) = (q(t), p(t)) \in C^\infty(S^1, T^*M)$, where $\lambda = p_i dq^i$ is the Liouville form.

We have the following correspondence:

$$q \text{ is a critical point of } \mathcal{L} \overset{1:1}{\leftrightarrow} x = (q,p),\ p = \partial_v L(t,q,\dot{q}) \text{ is a critical point of } \mathcal{A}.$$

By J. J. Dusitermaat [12], we also have:

$$\text{The Morse index of } \mathcal{L} \text{ at } q \in \mathrm{crit}(\mathcal{L}) = \text{the Maslov index of } x = (q,p) \in \mathrm{crit}(\mathcal{A}).$$

We do not mention about what the Maslov index is, but in our case it is the same as the relative Morse index after a normalization. We expect that the similar relation holds for our case: We expect

$$m_\lambda(\psi) = \text{the Morse index of a “dual action } \mathcal{L}^*\text{” at the dual critical point.}$$

Dual action

Assume that $H(x,\psi)$ is strictly convex in ψ. A natural candidate for a dual function is the Legendre-Fenchel dual $\mathcal{L}_H^*$ of H.

We consider our model example $H(x,\psi) = \frac{1}{p+1}H(x)|\psi|^{p+1}$. In this case, however, the dual action will not be C^2. Thus, we can not define the Morse index of the dual action. To define Morse index, however, the second order information of the functional is sufficient. Thus, we consider the dual action of the second order approximation $\mathcal{A}_{\psi,H}$ of $\mathcal{L}_H$. It is defined as

$$\begin{aligned}
\mathcal{A}_{\psi,H}(\varphi) &:= \frac{1}{2} d^2\mathcal{L}_H(\psi)(\varphi,\varphi) \\
&= \frac{1}{2}\int_M \langle \varphi, \mathsf{D}_{-\lambda}\varphi\rangle\, d\mathrm{vol}_g - \frac{\lambda}{2}\int_M |\varphi|^2\, d\mathrm{vol}_g - \frac{1}{2}\int_M H(x)|\psi|^{p-1}|\varphi|^2\, d\mathrm{vol}_g \\
&\quad - \frac{p-1}{2}\int_M H(x)|\psi|^{p-3}|\langle\psi,\varphi\rangle|^2\, d\mathrm{vol}_g \\
&= \frac{1}{2}\langle \mathsf{D}_{-\lambda}\varphi, \varphi\rangle_{H^{-1/2}\times H^{1/2}} - G_{H,\lambda}(\varphi),
\end{aligned}$$

where $G_{H,\lambda} : L^2(M, \mathbb{S}(M)) \to L^2(M, \mathbb{S}(M))$ is defined by

$$\begin{aligned}
G_{H,\lambda}(\varphi) &= \frac{\lambda}{2}\int_M |\varphi|^2\, d\mathrm{vol}_g + \frac{1}{2}\int_M H(x)|\psi|^{p-1}|\varphi|^2\, d\mathrm{vol}_g \\
&\quad + \frac{p-1}{2}\int_M H(x)|\psi|^{p-3}|\langle\psi,\varphi\rangle|^2\, d\mathrm{vol}_g.
\end{aligned}$$

The Legendre-Fenchel dual of $G_{H,\lambda}$ is defined by

$$G^*_{H,\lambda}(\varphi) := \max\{\langle\phi,\varphi\rangle_{L^2\times L^2} - G_{H,\lambda}(\phi) : \phi \in L^2(M,\mathbb{S}(M))\}.$$

An easy computation shows that

$$G^*_{H,\lambda}(\varphi) = \frac{1}{2}\int_M \frac{1}{\lambda + pH(x)|\psi|^{p-1}}|P_\psi(\varphi)|^2\, d\mathrm{vol}_g + \frac{1}{2}\int_M \frac{1}{\lambda + H(x)|\psi|^{p-1}}|P_\psi^\perp(\varphi)|^2\, d\mathrm{vol}_g,$$

where $P_\psi, P_\psi^\perp \in L^\infty(M, \mathrm{Sym}(\mathbb{S}(M))$ are defined as $P_\psi(x) =$ the orthogonal projection on $\langle\psi(x)\rangle$ and $P_\psi^\perp = \mathbf{1}_{\mathbb{S}(M)} - P_\psi$, respectively.

The dual action $\mathcal{A}^*_{\psi,H,\lambda}$ of $\mathcal{A}_{\psi,H}$ is defined as

$$\begin{aligned}
\mathcal{A}^*_{\psi,H,\lambda}(\varphi) &= G^*_H(\varphi) - \frac{1}{2}\langle K_\lambda\varphi,\varphi\rangle_{L^2\times L^2}\\
&= \frac{1}{2}\int_M \frac{1}{\lambda + pH(x)|\psi|^{p-1}}|P_\psi(\varphi)|^2\, d\mathrm{vol}_g + \frac{1}{2}\int_M \frac{1}{\lambda + H(x)|\psi|^{p-1}}|P_\psi^\perp(\varphi)|^2\, dvol_g\\
&\quad - \frac{1}{2}\int_M \langle K_\lambda\varphi,\varphi\rangle\, d\mathrm{vol}_g,
\end{aligned}$$

where $K_\lambda = \mathsf{D}_{-\lambda}^{-1}$ $(-\lambda \notin \mathsf{Spec}(\mathsf{D}_g))$.

We have:

Theorem 5 (Index formula I) *Assume $\lambda > 0$, $-\lambda \notin \mathsf{Spec}(\mathsf{D}_g)$. We have*

$$m_{-\lambda}(\psi) = \textit{the Morse index of } \mathcal{A}^*_{\psi,H,\lambda}.$$

As we will see shortly, $\mathcal{A}^*_{\psi,H,\lambda}$ behaves badly along the blow-up sequence $\psi = \psi_n$ defined in the previous subsection: The terms $\lambda + pH(x)|\psi|^{p-1}$ and $\lambda + H(x)|\psi|^{p-1}$ in $\mathcal{A}^*_{\psi,H,\lambda}$ are not pleasant for our purposes. In the presence of these, the continuity property (I-3) is difficult to obtain with the formula of Theorem 5.

To remedy this, we observe that $\mathcal{A}^*_{\psi,H,\lambda}$ is written as

$$\mathcal{A}^*_{\psi,H,\lambda} = ((H(x)|\psi|^{p-1} + \lambda L_\psi)^{-1}L_\psi(\varphi),\varphi)_{L^2} - (K_\lambda(\varphi),\varphi)_{L^2},$$

where $L_\psi = \frac{1}{p}P_\psi + P_\psi^\perp$.

After multiplication by $(H(x)|\psi|^{p-1} + \lambda L_\psi)^{1/2}$, we have the following similarity relation:

$$\mathcal{A}^*_{\psi,H,\lambda} \overset{\text{after multiplication}}{\cong} L_\psi - T_{\psi,H,\lambda},$$

where $T_{\psi,H,\lambda} = (H(x)|\psi|^{p-1} + \lambda L_\psi)^{1/2} \circ K_\lambda \circ (H(x)|\psi|^{p-1} + \lambda L_\psi)^{1/2}$. We thus arrive at the following:

Theorem 6 (Index formula II) *Assume $\lambda \geq 0$, $-\lambda \notin \mathsf{Spec}(\mathsf{D}_g)$.*

$$m_{-\lambda}(\psi) = \#\{\mu \in \mathsf{Spec}(L_\psi^{-1}T_{\psi,H,\lambda}) : \mu > 1\}.$$

Idea of the proof: For $0 \leq \theta \leq 1$, define

$$A_{\lambda,\psi,\theta} := \theta[(H(x)|\psi|^{p-1} + \lambda)\mathbf{1} + (p-1)H(x)|\psi|^{p-1}P_\psi],$$

We have a correspondence

$$\mathsf{D}_{-\lambda} - A_{\lambda,\psi,\theta} \overset{\text{L-F+multiplication trans.}}{\Leftrightarrow} \theta L_\psi^{-1} T_{\psi,H,\lambda} - \mathbf{1}.$$

We compare changes of the spectrum of the operators $\mathsf{D}_{-\lambda} - A_{\lambda,\psi,\theta}$ and $\theta L_\psi^{-1} T_{\psi,H,\lambda} - \mathbf{1}$ as θ changes form 0 to 1. For details, see [18]. □

4.2 Definition of $m_{\mathbb{R}^m}$

Motivated by the formula of Theorem 6, for $\psi \in L^\infty(M, \mathbb{S}(M))$, we define

$$T_\psi = |\psi|^{\frac{p-1}{2}} \circ \mathsf{D}_{g_{\mathbb{R}^m}}^{-1} \circ |\psi|^{\frac{p-1}{2}},$$

where $\mathsf{D}_{g_{\mathbb{R}^m}}^{-1} = -\omega_{m-1}^{-1} \frac{(x-y)}{|x-y|^m} \cdot$ is the Green kernel of $\mathsf{D}_{g_{\mathbb{R}^m}}$.

The integral representation of T_ψ is given as

$$(T_\psi \varphi)(x) = -\frac{1}{\omega_{m-1}} |\psi|^{\frac{p-1}{2}}(x) \int_{\mathbb{R}^m} \frac{(x-y)}{|x-y|^m} \cdot (|\psi|^{\frac{p-1}{2}}(y)\varphi(y))\, d\mathrm{vol}_{g_{\mathbb{R}^m}}(y).$$

Note that $T_\psi : L^2(\mathbb{R}^m, \mathbb{S}(\mathbb{R}^m)) \to L^2(\mathbb{R}^m, \mathbb{S}(\mathbb{R}^m))$ is in general an unbounded operator. We list below basic properties of T_ψ. For the proof, see [18].

(1) T_ψ is a densely defined self-adjoint operator on $L^2(\mathbb{R}^m)$ when $m \geq 3$.

(2) Its domain $\mathcal{D}(T_\psi)$ contains L^∞-spinors with compact supports when $m \geq 3$.

Definition 3 *Let $\psi \in L^\infty(\mathbb{R}^m, \mathbb{S}(\mathbb{R}^m))$ be a non-trivial solution to $\mathsf{D}_{g_{\mathbb{R}^m}}\psi = |\psi|^{p-1}\psi$ on $\mathbb{R}^m$. We define relative Morse index $m_{\mathbb{R}^m}(\psi)$ as the dimension of the maximal subspace of $\mathcal{D}(T_\psi)$ on which the following holds*

$$\frac{(T_\psi(\varphi), \varphi)_{L^2(\mathbb{R}^m)}}{(L_\psi(\varphi), \varphi)_{L^2(\mathbb{R}^m)}} > 1.$$

With this definition, properties (I-1), (I-2) in §3.4 are consequences of Theorem 6. The proof of (I-3) requires a uniform estimate of the Schwartz kernel of $\mathsf{D}_{g_n}^{-1}$ along the blowing-up manifolds $(M, g_n) \to (\mathbb{R}^m, g_{\mathbb{R}^m})$. For details, see [18].

4.3 Outline of the proof of (I-4)(Theorem 4)

The proof of (I-4) (Theorem 4) is based on the following:

Theorem 7 *Assume that $m \geq 2$ and $1 < p < \frac{m+1}{m-1}$. Let $\psi \in L^{p+1}(\mathbb{R}^m, \mathbb{S}(\mathbb{R}^m))$ be a solution to $\mathsf{D}_{g_{\mathbb{R}^m}}\psi = |\psi|^{p-1}\psi$ on $\mathbb{R}^m$. Then we have $\psi \equiv 0$.*

Sketch of the proof of (I-4): We observe that

$$d^2\mathfrak{L}(\psi)(\psi,\psi) = (1-p)\int_{\mathbb{R}^m} |\psi|^{p+1}\, d\mathrm{vol}_{g_{\mathbb{R}^m}} < 0$$

for $\mathfrak{L}(\varphi) = \frac{1}{2}\int_{\mathbb{R}^m}\langle \varphi, \mathsf{D}_{g_{\mathbb{R}^m}}\varphi\rangle\, d\mathrm{vol}_{g_{\mathbb{R}^m}} - \frac{1}{p+1}\int_{\mathbb{R}^m}|\psi|^{p+1}\, d\mathrm{vol}_{g_{\mathbb{R}^m}}$. (However, this calculation is not correct by Theorem 7!). By this (incorrect) calculation, a candidate for eigenspinor of $L_\psi^{-1}T_\psi$ with eigenvalue larger than 1 is given by $\mathsf{D}_{g_{\mathbb{R}^m}}\psi = |\psi|^{p-1}\psi$ multiplied by $|\psi|^{-\frac{p-1}{2}}$, i.e., $\varphi = |\psi|^{\frac{p-1}{2}}\psi$. In fact,

$$(T_\psi(\varphi),\varphi)_{L^2} = \int_{\mathbb{R}^m}|\psi|^{p+1}\, d\mathrm{vol}_{g_{\mathbb{R}^m}},$$

$$(L_\psi(\varphi),\varphi)_{L^2} = \frac{1}{p}\int_{\mathbb{R}^m}|\psi|^{p+1}\, d\mathrm{vol}_{g_{\mathbb{R}^m}}$$

and

$$\frac{(T_\psi(\varphi),\varphi)_{L^2}}{(L_\psi(\varphi),\varphi)_{L^2}} = p > 1.$$

(But, the calculation is not correct.)

To give a correct proof, we truncate φ suitably: Let $\eta \in C_0^\infty(\mathbb{R}^m)$ be such that $\eta(x) = 1$ for $|x| \leq 1$ and $\eta(x) = 0$ for $|x| \geq 2$. For $R > 0$, define $\eta_R(x) = \eta(x/R)$ and $\varphi_{\ell,R} = \eta_R^\ell |\psi|^{\frac{p-1}{2}}\psi$.

For large R and large ℓ ($\ell \geq \frac{p+1}{p-1}$ is sufficient), we can show that (see [18] for details)

$$\frac{(T_\psi(\varphi_{\ell,R}),\varphi_{\ell,R})_{L^2}}{(L_\psi(\varphi_{\ell,R}),\varphi_{\ell,R})_{L^2}} > 1.$$

By Theorem 7, ψ survives at infinity and suitably chosen sequence of cut-offs $R_1 < R_2 < \cdots < R_n < \cdots$ give arbitrary number of linearly independent φ_{ℓ,R_j} such that

$$\frac{(T_\psi(\varphi_{\ell,R_j}),\varphi_{\ell,R_j})_{L^2}}{(L_\psi(\varphi_{\ell,R_j}),\varphi_{\ell,R_j})_{L^2}} > 1.$$

The assertion of Theorem 4 follows from this. □

5 Morse-Floer homology $HF_*(\mathcal{L}_H, \mathcal{H}^{1/2}(M))$

In a recent work, Maalaoui [20] constructed Rabinowitz-Floer homology ([11]) for Dirac equations. It is a Floer homology for Lagrangian multiplier functional and it may be considered as a way of defining Floer homology for pure spinor action functional $\int_M \langle \psi, \mathsf{D}_g \psi \rangle \, d\mathrm{vol}_g$ on manifold $\{\psi : \int_M H(x, \psi) = 1\}$. However, for the present author, some of his arguments and assertions are difficult to understand. It seems that some more additional arguments are necessary to verify his assertions.

In any way, we consider "free" action functional and give an outline of the construction and the computation of the Morse-Floer homology of $\mathcal{H}^{1/2}(M)$ associated to $\mathcal{L}_H$ under assuming (2.2) and (2.3). In fact, it is not necessary to assume (2.2). We only need weaker condition

$$|H_\psi(x, \psi)| \leq C(1 + |\psi|^p). \tag{5.1}$$

Assume $\mathcal{L}_H$ is Morse. This is a generic condition for H, see [18] for the proof. We also assume $\mathbb{F} = \mathbb{Z}_2$ for simplicity. The case $\mathbb{F} = \mathbb{Z}$ will be treated in [18].

Recall that the graded group $\{C_p(\mathcal{L}_H, \mathcal{H}^{1/2}(M))\}_{p \in \mathbb{Z}}$ is defined as

$$C_p(\mathcal{L}_H, \mathcal{H}^{1/2}(M)) = \bigoplus_{\psi \in \mathrm{crit}_p(\mathcal{L}_H)} \mathbb{Z}_2 \langle \psi \rangle,$$

where $\mathrm{crit}_p(\mathcal{L}_H, \mathcal{H}^{1/2}(M)) = \{\psi \in \mathrm{crit}(\mathcal{L}_H) : m_\lambda(\psi) = p\}$.

We next give the definition of the boundary operator $\partial_p : C_p(\mathcal{L}_H, \mathcal{H}^{1/2}(M)) \to C_{p-1}(\mathcal{L}_H, \mathcal{H}^{1/2}(M))$. Let $\mathsf{x}, \mathsf{y} \in \mathrm{crit}(\mathcal{L}_H) := \{\psi \in \mathcal{H}^{1/2}(M) : d\mathcal{L}_H(\psi) = 0\}$. Let $\psi_0 \in C^1(\mathbb{R}, \mathcal{H}^{1/2}(M))$ be such that $\psi_0(t) = \mathsf{x}$ for $t \leq -1$, $\psi_0(t) = \mathsf{y}$ for $t \geq 1$. Define the trajectory space connecting x and y by

$$\mathcal{M}(\mathsf{x}, \mathsf{y}) = \left\{ \psi \in \psi_0 + W^{1,2}(\mathbb{R}, \mathcal{H}^{1/2}(M)) : \frac{\partial \psi}{\partial t} = -\nabla_{1/2} \mathcal{L}_H(\psi), \quad \psi(-\infty) = \mathsf{x}, \, \psi(+\infty) = \mathsf{y} \right\}.$$

Note that $\mathbb{R}$ acts freely on $\mathcal{M}(\mathsf{x}, \mathsf{y})$ via the time shift and we obtain the moduli space $\hat{\mathcal{M}}(\mathsf{x}, \mathsf{y}) = \mathcal{M}(\mathsf{x}, \mathsf{y})/\mathbb{R}$ of unparametrized trajectories connecting x and y.

$\mathcal{M}(\mathsf{x}, \mathsf{y})$ and hence $\hat{\mathcal{M}}(\mathsf{x}, \mathsf{y})$ are manifolds if 0 is a regular value of Fredholm map

$$\mathcal{F}_H : W^{1,2}(\mathbb{R}, \mathcal{H}^{1/2}(M)) \ni \psi \mapsto \frac{\partial \psi}{\partial t} + \nabla \mathcal{L}_H(\psi) \in L^2(\mathbb{R}, \mathcal{H}^{1/2}(M)).$$

This is equivalent to the condition that $W^u(\mathsf{x})$ and $W^s(\mathsf{y})$ intersect transversally at $\psi(t)$ for some $t \in \mathbb{R}$ (and hence for all $t \in \mathbb{R}$), where the unstable manifold $W^u(\mathsf{x})$ and the stable manifold $W^s(\mathsf{y})$ are defined by

$$W^u(\mathsf{x}) = \{z \in \mathcal{H}^{1/2}(M) : \lim_{t \to \infty} \psi(t, z) = x\}, \quad W^s(\mathsf{y}) = \{z \in \mathcal{H}^{1/2}(M) : \lim_{t \to +\infty} \psi(t, z) = y\},$$

where $\psi(t, \mathsf{z})$ is the solution to $\frac{\partial \psi}{\partial t} + \nabla \mathcal{L}_H(\psi) = 0$, $\psi(0, \mathsf{z}) = \mathsf{z}$.

Recall that the negative gradient flow $\psi(t, \cdot)$ is Morse-Smale if $W^u \mathsf{x})$ and $W^u(\mathsf{y})$ intersect transversally for any pair $\mathsf{x}, \mathsf{y} \in \mathrm{crit}(\mathcal{L}_H)$. In this case, $\hat{\mathcal{M}}(\mathsf{x}, \mathsf{y})$ is a manifold of $\dim \hat{\mathcal{M}}(\mathsf{x},\mathsf{y}) = m_\lambda(\mathsf{x}) - m_\lambda(\mathsf{y}) - 1$. We assume that $\mathcal{L}_H$ is Morse-Smale on $\mathcal{H}^{1/2}(M)$. (As we will explain shortly, this condition is in general never satisfied for $\mathcal{L}_H$ on $\mathcal{H}^{1/2}(M)$, however).

Under the assumption, we have:

• $\dim \hat{\mathcal{M}}(\mathsf{x},\mathsf{y}) = 0$ when $\mathsf{x} \in \mathrm{crit}_p(\mathcal{L}_H)$, $\mathsf{y} \in \mathrm{crit}_{p-1}(\mathcal{L}_H)$.

Furthermore, if $\hat{\mathcal{M}}(\mathsf{x},\mathsf{y})$ is compact, $\hat{\mathcal{M}}(\mathsf{x},\mathsf{y})$ consists of a finite number of points and we define

$$\partial_p \mathsf{x} = \sum_{\mathsf{y} \in \mathrm{crit}_{p-1}(\mathcal{L}_H)} n(\mathsf{x},\mathsf{y})\mathsf{y}, \tag{5.2}$$

where $n(\mathsf{x},\mathsf{y}) = \#(\hat{\mathcal{M}}(\mathsf{x},\mathsf{y})) \pmod 2$

To prove the boundary property $\partial_p \partial_{p-1} = 0$, we need $\hat{\mathcal{M}}(\mathsf{x},\mathsf{z})$ for $\mathsf{x},\mathsf{z} \in \mathrm{crit}(\mathcal{L}_H)$ with $m_\lambda(\mathsf{x}) - m_\lambda(\mathsf{y}) = 2$ and prove

$$\partial_p \partial_{p-1} \mathsf{x} = \sum_{\mathsf{z} \in \mathrm{crit}_{p-2}(\mathcal{L}_H)} \sum_{\mathsf{y} \in \mathrm{crit}_{p-2}(\mathcal{L}_H)} n(\mathsf{x},\mathsf{y}) n(\mathsf{y},\mathsf{z})\mathsf{z} = 0. \tag{5.3}$$

Thus, we want $\mathcal{M}(\mathsf{x},\mathsf{y})$ to be manifold for $\mathsf{x},\mathsf{y} \in \mathrm{crit}(\mathcal{L}_H)$ with $m_\lambda(\mathsf{x}) - m_\lambda(\mathsf{y}) = 1$ and 2.

To obtain Morse-Smale property for generic H, there are some technical problems which are not present in 1-dimensional variational problems. (Hamiltonian systems on symplectic manifolds are typical ones). More precisely, we have the following problems to construct the Morse-Floer homology for our case:

• **Regularity**: We want $\mathcal{L}_H$ is Morse for a generic H. We also want for a generic H and a generic metric on $\mathcal{H}^{1/2}(M)$, gradient flows are Morse-Smale at least for $\mathsf{x},\mathsf{y} \in \mathrm{crit}(\mathcal{L}_H)$ with $m_\lambda(\mathsf{x}) - m_\lambda(\mathsf{y}) \le 2$.

Compactness: We want $\hat{\mathcal{M}}(\mathsf{x},\mathsf{y})$ to be precompact and has a natural compactification.

However, they conflict to each other:

• It is possible to prove that $\mathcal{L}_H$ is Morse for generic H. To obtain the Morse-Smale property as stated above, we need at least C^3-regularity for $\mathcal{L}_H$ on $\mathcal{H}^{1/2}(M)$. This is the regularity versus Fredholm index assumption required for the use of Sard-Smale theorem. However, $\mathcal{L}_H$ is at most C^2 on $\mathcal{H}^{1/2}(M)$ even if we assume $H \in C^\infty$. To remedy the lack of regularity which occurs when we are working on $\mathcal{H}^{1/2}(M)$, we need to work on more regular spinor space.

• On the other hand, compactness is easier to obtain when working on less regular spinor space like $\mathcal{H}^{1/2}(M)$.

Abbondandolo and Majer [2], [3], [4] constructed general Morse-Floer theory for a class of strongly indefinite functional defined on Hilbert manifolds. However, their general theory does not directly applicable due to the above problems. In a fundamental work of [6], they constructed Morse-Floer homology for elliptic systems.

In that problem, similar problems also raised. Our construction of the Morse-Floer homology of $\mathcal{H}^{1/2}(M)$ associated to $\mathcal{L}_H$ owes much to their ideas.

Under **subcritical** and **superquadratic** conditions on H, together with the ellipticity of D_g and its nice mapping properties in various function spaces, instead of working with $\mathcal{H}^{1/2}(M)$, we can work with more regular spinor space $\mathcal{C}^{0,\alpha}(M) := C^{0,\alpha}(M, \mathbb{S}(M))$ (for $1/2 < \alpha < 1$) and resolve both problems at the same time. That is, we have:

(1) $\mathcal{L}_H$ is C^k on $\mathcal{C}^{0,\alpha}(M)$ ($\frac{1}{2} < \alpha < 1$) if H is C^{k+1}.

(2) $\mathrm{crit}(\mathcal{L}_H) \subset \mathcal{C}^{0,\alpha}(M)$:

(3) $\mathcal{C}^{0,\alpha}(M)$ is invariant under the $H^{1/2}$-gradient flow.

(4) $\mathcal{L}_H$ satisfies the Palais-Smale condition on $\mathcal{H}^{1/2}(M)$.

(5) $\psi \in \mathcal{M}(x, y)$ is uniformly bounded in $\mathcal{C}^{0,\alpha}(M)$; $\sup_{t\in\mathbb{R}} \|\psi(t)\|_{C^{0,\alpha}} < +\infty$.

(1)–(3) are regularity conditions, while (4) and (5) are compactness conditions.

There are also technical issues to overcome. In anyway, from these, we have (see [18] for details)

(1) $\mathcal{L}_H$ is Morse for generic H.

(2) For $\mathsf{x}, \mathsf{y} \in \mathrm{crit}(\mathcal{L}_H)$ with $m_\lambda(\mathsf{x}) - m_\lambda(\mathsf{y}) \leq 2$, $\hat{\mathcal{M}}(\mathsf{x}, \mathsf{y})$ is a manifold of dimension $m_\lambda(\mathsf{x}) - m_\lambda(\mathsf{y}) - 1$ for generic H and generic metric on $\mathcal{H}^{1/2}(M)$.

(3) $\hat{\mathcal{M}}(\mathsf{x}, \mathsf{y}) \subset C^0_{\mathrm{loc}}(\mathbb{R}, \mathcal{C}^{0,\alpha}(M))$ is precompact and has a natural compactification $\overline{\hat{\mathcal{M}}}(\mathsf{x}, \mathsf{y})$ for $\mathsf{x}, \mathsf{y} \in \mathrm{crit}(\mathcal{L}_H)$ with $m_\lambda(\mathsf{x}) - m_\lambda(\mathsf{y}) \leq 2$.

Also, it is easy to see that $\partial\overline{\hat{\mathcal{M}}}(\mathsf{x}, \mathsf{y})$ is a gradient flow invariant compact set. Thus, if $\mathcal{L}_H$ is Morse and the gradient flow is Morse-Smale, then $\hat{\mathcal{M}}(\mathsf{x}, \mathsf{y})$ consists of a finitely many critical points of $\mathcal{L}_H$ and connecting orbits of these. From these, we have:

• $\hat{\mathcal{M}}(\mathsf{x}, \mathsf{y})$ is compact when $m(\mathsf{x}) - m(\mathsf{y}) = 1$ and $\partial_p : C_p(\mathcal{L}_H, \mathcal{H}^{1/2}(M)) \to C_p(\mathcal{L}_H, \mathcal{H}^{1/2}(M))$ defined by the formula (5.2) is well-defined.

• $\hat{\mathcal{M}}(\mathsf{x}, \mathsf{z})$ is a 1-dimentional manifold with boundary when $m_\lambda(\mathsf{x}) - m_\lambda(\mathsf{z}) = 2$. $\partial\hat{\mathcal{M}}(\mathsf{x}, \mathsf{z})$ consists precisely of "1-breaking" orbits (this requires gluing construction which we skipped in the above argument).

In the formula of $\partial_{p-1}\partial_p$ in (5.3), the matrix element

$$\sum_{\mathsf{y} \in \mathrm{crit}_{p-1}(\mathcal{L}_H)} n(\mathsf{x}, \mathsf{y}) n(\mathsf{y}, \mathsf{z})$$

counts the number of the connected components of $\partial\hat{\mathcal{M}}(\mathsf{x}, \mathsf{z})$ which is even and 0 (mod 2). This proves the boundary property: $\partial_{p-1}\partial_p = 0$. In this way, we have a well-defined homology of the chain complex $\{C_p(\mathcal{L}_H, \mathcal{H}^{1/2}(M)), \partial_p\}_{p\in\mathbb{Z}}$:

$$HF_p(\mathcal{L}_H, \mathcal{H}^{1/2}(M); \mathbb{Z}_2) := \ker \partial_p / \mathrm{Im}\, \partial_{p+1}$$

for generic H and generic metric on $\mathcal{H}^{1/2}(M)$.

To define $HF(\mathcal{L}_H, \mathcal{H}^{1/2}(M); \mathbb{Z}_2)$ for general H, we take a generic H' which satisfies (2.1), (2.2) and $\|H - H'\|_{L^\infty} < \epsilon$ and a generic metric on $\mathcal{H}^{1/2}(M)$ such that $\mathcal{L}_{H'}$ is Morse and the negative gradient flow system is Morse-Smale. Then $HF_*(\mathcal{L}_{H'}, \mathcal{H}^{1/2}(M); \mathbb{Z}_2)$ is defined (we omitted to indicate the dependence of the metric on $\mathcal{H}^{1/2}(M)$, but $HF_*(\mathcal{L}_H, \mathcal{H}^{1/2}(M); \mathbb{Z}_2)$ defined so far indeed depends on the choice of the metric on $\mathcal{H}^{1/2}(M)$).

The next step is to show that $HF_*(\mathcal{L}_{H'}, \mathcal{H}^{1/2}(M); \mathbb{Z}_2)$ does not depend on the choices of a generic H' and a generic metric on $\mathcal{H}^{1/2}(M)$. Note that, in general, the Morse homology on a non-compact manifold depends on a chosen Morse function. This is in contrast to the compact case. Thus, to obtain stability result, we need a restriction on a class of functions. A general result is stated as follows:

Theorem 8 *For generic pairs H_0, G_0 and H_1, G_1 (H_0, H_1 are generic functions satisfying (2.1), (2.2) and G_0, G_1 are generic metrics on $\mathcal{H}^{1/2}(M)$), we have a natural isomorphism*

$$HF_*(\mathcal{L}_{H_0}, (\mathcal{H}^{1/2}(M), G_0); \mathbb{Z}_2) \cong HF_*(\mathcal{L}_{H_1}, (\mathcal{H}^{1/2}(M), G_1); \mathbb{Z}_2)$$

provided $\|H_0 - H_1\|_{L^\infty(\mathbb{S}(M))} < +\infty$.

By Theorem 8, we can define

$$HF_*(\mathcal{L}_H, \mathcal{H}^{1/2}(M)) = HF_*(\mathcal{L}_{H'}, (\mathcal{H}^{1/2}(M), G'))$$

for a generic H' and G' which satisfies $\|H - H'\|_{L^\infty(\mathbb{S}(M)} < +\infty$.

Outline of the proof of Theorem 8: The proof is also standard, see [13], [6]. Let $\rho \in C^\infty(\mathbb{R})$ be such that $\rho(t) = 0$ for $t \le -1$ and $\rho(t) = 1$ for $t \ge 1$. We consider time dependent function and metric which interpolate between H_0 and H_1 and G_0 and G_1, respectively:

$$H_{1,0}(t, x, \psi) = (1 - \rho(t))H_0(x, \psi) + \rho(t)H_1(x, \psi),$$

$$G_{1,0}(t, \psi) = (1 - \rho(t))G_0(\psi) + \rho(t)G_1(\psi).$$

We consider non-autonomous system

$$\frac{\partial \psi}{\partial t} = -\nabla_{G_{1,0}}\mathcal{L}_{H_{1,0}}(\psi), \quad \psi(-\infty) = \mathrm{x}_0, \quad \psi(+\infty) = \mathrm{x}_1,$$

where $\mathrm{x}_0 \in \mathrm{crit}_p(\mathcal{L}_{H_0})$, $\mathrm{x}_1 \in \mathrm{crit}_q(\mathcal{L}_{H_1})$ and $\nabla_{G_{1,0}}\mathcal{L}_{H_{1,0}}$ is the gradient of $\mathcal{L}_{H_{1,0}}$ with respect to the metric $G_{1,0}$.

We consider the moduli space of solutions to the above system $\mathcal{M}_{H_{1,0},G_{1,0}}(\mathrm{x}_0, \mathrm{x}_1)$. Under the assumption, $H_{1,0}$ satisfies (2.1) and (2.2) uniformly for $t \in \mathbb{R}$ and we can show, as in the autonomous case, $\mathcal{M}_{H_{1,0},G_{1,0}}(\mathrm{x}_0, \mathrm{x}_1)$ is precompact in $C^0_{\mathrm{loc}}(\mathbb{R}, \mathcal{C}^{0,\alpha}(M))$ and it has a natural compactification whose boundary consists of broken trajectories.

After perturbing $H_{1,0}$ and $G_{1,0}$ if necessary, $\mathcal{M}_{H_{1,0},G_{1,0}}(\mathrm{x}_0,\mathrm{x}_1)$ is a manifold of dimension $m_\lambda(\mathrm{x}_0)-m_\lambda(\mathrm{x}_1)$. In particular, for the case $m_\lambda(\mathrm{x}_0)=m_\lambda(\mathrm{x}_1)$, $\mathcal{M}_{H_{1,0},G_{1,0}}(\mathrm{x}_0,\mathrm{x}_1)$ is compact and we can define

$$\Phi : C_p(\mathcal{L}_{H_0},\mathcal{H}^{1/2}(M)) \to C_p(\mathcal{L}_{H_1},\mathcal{H}^{1/2}(M))$$

by counting trajectories:

$$\Phi_{1,0}(\mathrm{x}_0)=\sum_{\mathrm{x}_1\in \mathrm{crit}_p(\mathcal{L}_{H_1})} n_{H_{1,0},G_{1,0}}(\mathrm{x}_0,\mathrm{x}_1)\mathrm{x}_1,$$

where $n_{H_{1,0},G_{1,0}}(\mathrm{x}_0,\mathrm{x}_1)=\#\mathcal{M}_{H_{1,0},G_{1,0}}(\mathrm{x}_0,\mathrm{x}_1) \pmod 2$.

By examining the boundary $\partial\mathcal{M}_{H_{1,0},G_{1,0}}(\mathrm{x}_0,\mathrm{x}_1)$ for $\mathrm{x}_0 \in \mathrm{crit}(\mathcal{L}_{H_0})$ and $\mathrm{x}_1 \in \mathrm{crit}(\mathcal{L}_{H_1})$ with $m_\lambda(\mathrm{x}_0)-m_\lambda(\mathrm{x}_1)=1$, we see that $\Phi_{1,0}$ is a chain map: $\partial_{H_1}\circ\Phi_{1,0}+\Phi_{1,0}\circ\partial_{H_0}=0$. Moreover, considering homotopies of homotopies, we see that $\Phi_{1,0}$ is natural in the sense that

$$\Phi_{0,0}=\mathbf{1},\quad \Phi_{2,1}\circ\Phi_{1,0}=\Phi_{2,0}.$$

These imply that $\Phi_{1,0}$ induces the isomorphism of homologies:

$$\Phi_{1,0} : HF_*(\mathcal{L}_{H_0},(\mathcal{H}^{1/2}(M),G_0)) \cong HF_*(\mathcal{L}_{H_1},(\mathcal{H}^{1/2}(M),G_1)).$$

□

Based on the above Theorem 8, we have

Theorem 9 *Assume that $H\in C^2(\mathbb{S}(M))$ satisfies (2.1) and (2.2). Then the Morse-Floer homology $FH_*(\mathcal{L}_H,\mathcal{H}^{1/2}(M);\mathbb{Z}_2)$ is well-defined. For the case $H(x,\psi)=\frac{1}{p+1}H(x)|\psi|^{p+1}$ (more generally, for H satisfying "strong superquadratic condition at infinity"), we have a vanishing result $HF_*(\mathcal{L}_H,\mathcal{H}^{1/2}(M))=0$.*

For details, see [18].

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